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# EFFECTIVE STIFFNESS THEORY FOR A LAMINATED ELASTIC–VISCOPLASTIC WORK-HARDENING COMPOSITE

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**Abstract**—Effective stiffness theory is derived for modeling the dynamic behavior of a laminated composite material made of elastic–viscoplastic work-hardening constituents. The resulting theory represents the composite as a higher order continuum with microstructure, with the corresponding equations of motion and flow rules. The theory is illustrated and applied to obtain the dynamic response of a laminated slab subjected to time-dependent normal extended load in the direction of the layering.

## INTRODUCION

Considerable work has been done to construct continuum theories modeling the dynamic behavior of composite materials. The well known equivalent modulus theory replaces the composite by an equivalent homogeneous anisotropic solid which is characterized by its effective moduli, see Hashin[1] and the recent book by Christensen[2] for example. The basic deficiency of this theory is its inability to reflect the microstructure effects of the composite and thus it does not exhibit wave dispersion and attenuation observed in the multiphase material.

In a more elaborated theory, the composite material must be represented as a higher order continuum with microstructure and thus a more realistic description is achieved. One type of such a theory is the effective stiffness theory which for a laminated elastic materials was presented in[3]. This theory was further developed and the reader is referred to the monograph by Achenbach[4] and the review article by Bedford *et al.*[5] for further details.

In particular, Grot and Achenbach extended the theory to a visco-elastic anisothermal[6] and nonlinear perfectly elastic[7] laminated composite. The latter theory was implemented in[8] where a comparison between the dynamic response based on the effective stiffness theory and the exact theory of elasticity was also given.

Essential difficulties arise when the laminated medium is made of inelastic constituents due to the non-linear constitutive equations and their dependence on the history of deformation being path-dependent (loading or unloading). These constitutive equations can be given only as relationships between strain and stress rates. Consequently, whereas the distribution of strains across a layer thickness can be assumed to be linear as in[6], the distribution of stresses cannot be assumed to be linear and in fact higher order terms are necessary for representing the stresses across the thickness of a layer. Similar difficulties are met in the theory of inelastic shells where Wempner and Hwang[9] employed recently the Legendre polynomials to approximate the distribution of stresses in the section of an inelastic shell.

In this paper we propose an effective stiffness theory for elastic–viscoplastic laminated medium in which the stress distribution across the layers are represented as in[9] by higher order terms. Every constituent is assumed to be elastic–viscoplastic material which is represented by the unified theory of Bodner and Partom[10] in which no yield criterion or loading or unloading conditions are required. This strain-rate theory includes also isotropic work-hardening and in its basic formulation the material is characterized by five constants.

In the following sections the effective stiffness theory is developed and the field equations governing the deformation of the continuum are given. The theory is illustrated and applied for the problem of a laminated elastic–viscoplastic slab subjected to a time-dependent normal extended load in the laminae direction. The dynamic response of the slab is obtained by a numerical solution of the equations of motion and the flow rules for plastic strains and plastic work. Results are given illustrating the effect of the viscoplastic mechanism by comparison with the response of the corresponding perfectly elastic laminated slab. The latter is obtained as a special case of the present theory.

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## BASIC EQUATIONS FOR ELASTIC-VISCOPLASTIC MATERIALS

Consider a periodic array of two alternating plane layers made of isotropic elastic–viscoplastic work-hardening materials. Let $d_1$, $d_2$ denote the width of each layer. Local systems of Cartesian coordinates $(x_1, \bar{x}_2^{(\alpha)}, x_3)$ are introduced such that $\bar{x}_2^{(\alpha)}$ denotes the distance from the midplane of layer $\alpha$, see Fig. 1(a)†.

The constitutive equations of elastic–viscoplastic work-hardening materials adopted in this paper are those proposed by Bodner and Partom[10] which have the property that no yield criterion nor loading or unloading conditions are required. In this formulation both elastic and inelastic deformations are present at all stages of loading and unloading.

The constitutive equations of the material $\alpha$ can be described by separating the total strain rate components into elastic (reversible) and plastic (irreversible) strain rates as follows:

$$\dot{\epsilon}_{ij}^{(\alpha)} = \dot{\epsilon}_{ij}^{(e\alpha)} + \dot{\epsilon}_{ij}^{(p\alpha)} \qquad i, j = 1, 2, 3 \tag{1}$$

where $\epsilon_{ij}^{(\alpha)} = [\partial_j u_i^{(\alpha)} + \partial_i u_j^{(\alpha)}]/2$ with $u_i^{(\alpha)}$ being the components of the displacement vector (so that the strains are considered to be infinitesimal), dots represent time derivatives and $\partial_j u_i^{(\alpha)} = (\partial/\partial x_j) u_i^{(\alpha)}$ where differentiation in $x_2$—direction should be with respect to the local coordinate $\bar{x}_2^{(\alpha)}$ whenever a dependent variable is expressed in terms of local coordinates. The elastic strain rates $\dot{\epsilon}_{ij}^{(e\alpha)}$ are related to the stress rates $\dot{\sigma}_{ij}^{(\alpha)}$ according to the usual Hooke's law

$$\dot{\epsilon}_{ij}^{(e\alpha)} = \dot{\sigma}_{ij}^{(\alpha)}/(2\mu_\alpha) - (\nu_\alpha/E_\alpha)\dot{\sigma}_{kk}^{(\alpha)}\delta_{ij} \tag{2}$$

where $\mu_\alpha$, $\nu_\alpha$, $E_\alpha$ are, respectively, the rigidity, the Poisson ratio and the Young modulus of the material and $\delta_{ij}$ is the Kronecker delta.

The plastic strain rates $\dot{\epsilon}_{ij}^{(p\alpha)}$ are related to the stresses according to the flow rule

$$\dot{\epsilon}_{ij}^{(p\alpha)} = \dot{e}_{ij}^{(p\alpha)} = \Lambda_\alpha s_{ij}^{(\alpha)} \tag{3}$$

where $s_{ij}^{(\alpha)}$ and $\dot{e}_{ij}^{(p\alpha)}$ denote the deviators of the stress and plastic strain rate tensors respectively, i.e. $s_{ij}^{(\alpha)} = \sigma_{ij}^{(\alpha)} - \sigma_{kk}^{(\alpha)}\ \delta_{ij}/3$ and $\dot{e}_{ij}^{(p\alpha)} = \dot{\epsilon}_{ij}^{(p\alpha)} - \dot{\epsilon}_{kk}^{(p\alpha)}\ \delta_{ij}/3$. According to (3) the plastic deformations are necessarily incompressible, i.e. $\dot{\epsilon}_{kk}^{(p\alpha)} = 0$.

Equation (3) can be squared to obtain $\Lambda_\alpha$,

$$\Lambda_\alpha^{\ 2} = D_2^{(p\alpha)}/J_2^{(\alpha)} \tag{4}$$

where

$$D_2^{(p\alpha)} = \dot{e}_{ij}^{(p\alpha)}\dot{e}_{ij}^{(p\alpha)}/2, \quad J_2^{(\alpha)} = s_{ij}^{(\alpha)}s_{ij}^{(\alpha)}/2 \tag{5}$$

which are the second invariants of the plastic strain rate deviator and the stress deviator tensors, respectively. Motivated by equations relating dislocation velocities and stresses,

†Here and in the sequel the subscript or superscript $\alpha$ will indicate that quantities belong to either one of the constituents. Repeated $\alpha$ does not imply summation.

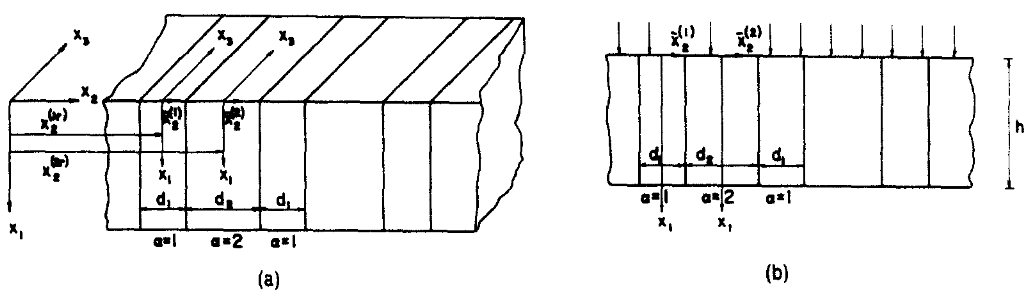

Fig. 1. (a) A laminated medium. (b) A laminated slab subjected to extended normal load.

Bodner and Partom[10] proposed the relation

$$D_2^{(p\alpha)} = D_0^{(\alpha)2} \exp\{-[Z_\alpha^2/3J_2^{(\alpha)}]^{n_\alpha}(n_\alpha+1)/n_\alpha\} \tag{6}$$

where $n_\alpha$ is related to the steepness of the $D_2^{(p\alpha)} - J_2^{(\alpha)}$ curve, $(D_0^{(\alpha)})^2$ is the limiting value of $D_2^{(p\alpha)}$ for very high stresses and $Z_\alpha$ is an internal state variable, referred to as the hardness of the material, which expresses its overall resistance to plastic flow. For isotropic work-hardening, the evolution equation for $Z_\alpha$ is taken to depend on the amount of plastic (irreversible) work $w_p^{(\alpha)}$ which has been done on the material from a reference state. Specifically, $Z_\alpha$ is assumed to have the form

$$Z_\alpha = Z_1^{(\alpha)} + \left(Z_0^{(\alpha)} - Z_1^{(\alpha)}\right) \exp[-m_\alpha w_p^{(\alpha)}/Z_0^{(\alpha)}] \tag{7}$$

where $Z_0^{(\alpha)}$, $Z_1^{(\alpha)}$ and $m_\alpha$ are appropriate parameters of the material, and the rate of plastic work can be expressed in the form

$$\dot{w}_p^{(\alpha)} = \sigma_{ij}^{(\alpha)}\dot{\epsilon}_{ij}^{(p\alpha)} = s_{ij}^{(\alpha)}\dot{\epsilon}_{ij}^{(p\alpha)} = 2\Lambda_\alpha J_2^{(\alpha)}. \tag{8}$$

In (7), $Z_0^{(\alpha)}$ is the initial hardness and $Z_1^{(\alpha)}$ is the upper limit of $Z_\alpha$ (saturation value) since otherwise, $D_2^{(p\alpha)}$ would approach zero for large $w_p^{(\alpha)}$ which leads to fully elastic behavior at appreciable strains.

Finally the stresses fulfill, in the absence of body forces, the usual equations of motion

$$\rho_\alpha \ddot{u}_i^{(\alpha)} = \partial_j \sigma_{ij}^{(\alpha)} \tag{9}$$

where $\rho_\alpha$ is the density of the material.

## DERIVATION OF THE EFFECTIVE STIFFNESS THEORY

Let $x_2^{(1r)}$, $x_2^{(2r)}$ denote the midplane positions of the $r$-th pair of layers, see Fig. 1(a). The essential step of the effective stiffness theory is the expansion of the displacement vector in each layer in terms of the distance from the layer centerline. This expansion can be expressed in terms of the Legendre polynomials permitting the modeling of increasingly complex deformation patterns in the layer. For a first order theory the displacement at any point within a layer can be approximated by[6]

$$u_i^{(\alpha)} = \bar{u}_i^{(\alpha r)}(x_1, x_2^{(\alpha r)}, x_3, t) + \bar{x}_2^{(\alpha)}\Psi_i^{(\alpha r)}(x_1, x_2^{(\alpha r)}, x_3, t) \tag{10}$$

where $\bar{u}_i^{(\alpha r)}$ are the displacement components in the midplane of the layer $\alpha$ in the $r$-th pair, $\Psi_1^{(\alpha r)}$, $\Psi_3^{(\alpha r)}$ represent antisymmetric thickness shear deformations and $\Psi_2^{(\alpha r)}$ represents symmetric thickness stretch deformation in the layer.

Grot and Achenbach[6] show that a homogeneous continuum model for the laminated medium can be constructed by replacing the discrete variables $x_2^{(\alpha r)}$ by the corresponding coordinate $x_2$ and considering the field variables $\bar{u}_i^{(\alpha r)}$, $\Psi_i^{(\alpha r)}$ as continuous functions of $x_2$. The transition from the discrete to the continuous field is indicated by writing $\bar{u}_i^{(\alpha)}(x_1, x_2, x_3, t)$ instead of $u_i^{(\alpha r)}(x_1, x_2^{(\alpha r)}, x_3, t)$ with similar notation for the other field variables.

The condition of the continuity of the displacements at the interface between the layers of the $r$-th pair takes the form[5]

$$\bar{u}_i^{(1)} = \bar{u}_i^{(2)}, \tag{11}$$

$$-d_1\partial_2\bar{u}_i^{(1)} + d_1\Psi_i^{(1)} = d_2\partial_2\bar{u}_i^{(2)} - d_2\Psi_i^{(2)}. \tag{12}$$

The first condition shows that the mid-plane displacement is given by the single function

$\bar{u}_i(x_j, t)$ (the gross displacement), and the second condition (12) can be simplified in the form

$$\partial_2 \bar{u}_i(x_j, t) = \theta \Psi_i^{(1)}(x_j, t) + (1-\theta)\Psi_i^{(2)}(x_j, t) \tag{13}$$

with $\theta = d_1/(d_1 + d_2)$.

The total strain rate $\dot{\epsilon}_{ij}^{(\alpha)}$ can be computed from the expansion (10) giving

$$\dot{\epsilon}_{ij}^{(\alpha)}(x_k, z^{(\alpha)}, t) = \dot{\eta}_{ij}^{(\alpha)}(x_k, t)P_0(z^{(\alpha)}) + \sqrt{(3)}\dot{\xi}_{ij}^{(\alpha)}(x_k, t)P_1(z^{(\alpha)}) \tag{14}$$

where $z^{(\alpha)} = 2\bar{x}_2^{(\alpha)}/d_\alpha$, $P_n$ are the Legendre polynomials and

$$\left.\begin{aligned} &\eta_{pq}^{(\alpha)} = (\partial_q \bar{u}_p + \partial_p \bar{u}_q)/2 && p, q = 1, 3 \\ &\eta_{2q}^{(\alpha)} = \eta_{q2}^{(\alpha)} = (\partial_q \bar{u}_2 + \Psi_q^{(\alpha)})/2 && q = 1, 3 \\ &\eta_{22}^{(\alpha)} = \Psi_2^{(\alpha)} \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} &\xi_{pq}^{(\alpha)} = c_\alpha(\partial_q \Psi_p^{(\alpha)} + \partial_p \Psi_q^{(\alpha)}) && p, q = 1, 3 \\ &\xi_{2q}^{(\alpha)} = \xi_{q2}^{(\alpha)} = c_\alpha \partial_q \Psi_2^{(\alpha)} && q = 1, 3 \\ &\xi_{22}^{(\alpha)} = 0. \end{aligned}\right\} \tag{16}$$

with $c_\alpha = d_\alpha/(4\sqrt{(3)})$.

In Ref. [6] balance equations for the linear momentum and for the moment of momentum of the homogeneous continuum model have been obtained by integrating the stresses in (9) and their first moments with respect to the layer thickness and making the previously mentioned transition from the system of discrete layers to the homogeneous continuum model. In the process of deriving these equations new quantities called "composite stresses", "interface stresses", "double stresses" and "layer stresses" were introduced.

With zero body forces we obtain as in [6]

$$\partial_p \bar{\sigma}_{pj} + \partial_2 \Sigma_{2j} = \rho \ddot{\bar{u}}_j \tag{17}$$

$$\partial_p M_{p2j}^{(\alpha)} + \Sigma_{2j} - \bar{\sigma}_{2j}^{(\alpha)} = I_\alpha \ddot{\Psi}_j^{(\alpha)} \tag{18}$$

where $p = 1, 3$ and $\bar{\sigma}_{pj}$ are the components of the composite stresses given by

$$(d_1 + d_2)\bar{\sigma}_{pj} = \mathrm{Int}^{(1)}[\sigma_{pj}^{(1)}] + \mathrm{Int}^{(2)}[\sigma_{pj}^{(2)}] \tag{19}$$

with the notation

$$\mathrm{Int}^{(\alpha)}[g^{(\alpha)}] = \int_{-d_\alpha/2}^{d_\alpha/2} g^{(\alpha)}(x_1, x_2, x_3, \bar{x}_2^{(\alpha)}, t)\, \mathrm{d}\bar{x}_2^{(\alpha)} \tag{20}$$

and $\rho = \theta\rho_1 + (1-\theta)\rho_2$ being the effective density. In (17) $\Sigma_{2j}(x_i, t)$ are the interface stresses defined in [6] and

$$\rho(d_1 + d_2)\ddot{\bar{u}}_j = \mathrm{Int}^{(1)}[\rho_1 \ddot{u}_j^{(1)}] + \mathrm{Int}^{(2)}[\rho_2 \ddot{u}_j^{(2)}]. \tag{21}$$

The double stresses and interface stresses in (18) are given respectively by

$$d_\alpha M_{p2j}^{(\alpha)} = \mathrm{Int}^{(\alpha)}[\bar{x}_2^{(\alpha)} \sigma_{pj}^{(\alpha)}] \tag{22}$$

$$d_\alpha \bar{\sigma}_{2j}^{(\alpha)} = \mathrm{Int}^{(\alpha)}[\sigma_{2j}^{(\alpha)}] \tag{23}$$

and $I_\alpha = \rho_\alpha d_\alpha^2/12$.

For elastic or viscoelastic materials once the stresses are expressed in terms of the displacement gradients according to the appropriate constitutive equation, the field eqns (17)–(18) give the equations of motion of the material in the framework of the effective stiffness theory. In contrast, the behavior of an elastic–plastic material depends upon the history of deformation being path-dependent and the constitutive equations are nonlinear and given in terms of the stress and strain rates as in eqns (1)–(3). Following Wempner and Hwang[9], let us represent the stress distribution in each layer in terms of the Legendre polynomials in the form

$$\sigma_{ij}^{(\alpha)}(x_k, z^{(\alpha)}, t) = (1+2n)^{1/2}\tau_{ij(n)}^{(\alpha)}(x_k, t)P_n(z^{(\alpha)}) \tag{24}$$

where

$$\tau_{ij(n)}^{(\alpha)} = [(1+2n)^{1/2}/2]\int_{-1}^{1}\sigma_{ij}^{(\alpha)}P_n(z^{(\alpha)})\,\mathrm{d}z^{(\alpha)} \tag{25}$$

and summation is implied by the repeated index $n = 0, 1, \ldots, N$.

The power of the stress in the layer is approximated by[9]

$$\dot{w}^{(\alpha)}(x_k, t) = \frac{1}{2}\int_{-1}^{1}\sigma_{ij}^{(\alpha)}\dot{\epsilon}_{ij}^{(\alpha)}\,\mathrm{d}z^{(\alpha)} \tag{26}$$

where $\dot{\epsilon}_{ij}^{(\alpha)}(x_k, z^{(\alpha)}, t)$ is given by (14). It follows from (24)–(25), (14) that

$$\dot{w}^{(\alpha)} = \tau_{ij(0)}^{(\alpha)}\dot{\eta}_{ij}^{(\alpha)} + \tau_{ij(1)}^{(\alpha)}\dot{\xi}_{ij}^{(\alpha)}. \tag{27}$$

Another form for $\dot{w}$ is obtained if (1)–(3) are used instead of (14):

$$\dot{w}^{(\alpha)} = \tau_{ij(n)}^{(\alpha)}[\dot{\tau}_{ij(n)}^{(\alpha)}/2\mu_\alpha - (\nu_\alpha/E_\alpha)\dot{\tau}_{kk(n)}^{(\alpha)}\delta_{ij} + t_{ij(m)}^{(\alpha)}F_{nm}^{(\alpha)}] \tag{28}$$

where $m = 0, 1, \ldots, N$,

$$t_{ij(n)}^{(\alpha)} = \tau_{ij(n)}^{(\alpha)} - \tau_{kk(n)}^{(\alpha)}\delta_{ij}/3 \tag{29}$$

and

$$F_{nm}^{(\alpha)}(x_k, t) = [(1+2n)(1+2m)]^{1/2}/2\int_{-1}^{1}P_n(z^{(\alpha)})P_m(z^{(\alpha)})\Lambda_\alpha\,\mathrm{d}z^{(\alpha)}. \tag{30}$$

The balance of (27) and (28) is satisfied by the equations

$$\dot{\eta}_{ij}^{(\alpha)} = (1/2\mu_\alpha)\dot{\tau}_{ij(0)}^{(\alpha)} - (\nu_\alpha/E_\alpha)\dot{\tau}_{kk(0)}^{(\alpha)}\delta_{ij} + t_{ij(m)}^{(\alpha)}F_{0m}^{(\alpha)} \tag{31}$$

$$\dot{\xi}_{ij}^{(\alpha)} = (1/2\mu_\alpha)\dot{\tau}_{ij(1)}^{(\alpha)} - (\nu_\alpha/E_\alpha)\tau_{kk(1)}^{(\alpha)}\delta_{ij} + t_{ij(m)}^{(\alpha)}F_{1m}^{(\alpha)} \tag{32}$$

$$0 = (1/2\mu_\alpha)\dot{\tau}_{ij(n)}^{(\alpha)} - (\nu_\alpha/E_\alpha)\tau_{kk(n)}^{(\alpha)}\delta_{ij} + t_{ij(m)}^{(\alpha)}F_{nm}^{(\alpha)} \tag{33}$$

$n = 2, 3, \ldots, N$.

It should be mentioned that the averaging integral (26) made it possible to assume a linear distribution for the strain rate across the layer (see eqn 14), and still requiring a nonlinear stress distribution.

If we define the $n$-th plastic strain rate in the form

$$\dot{L}_{ij(n)}^{(\alpha)}(x_k, t) = t_{ij(m)}^{(\alpha)}F_{nm}^{(\alpha)}, \tag{34}$$

we readily obtain by inverting (31)–(33) the relations

$$\dot{\tau}_{ij(0)}^{(\alpha)} = \lambda_\alpha\dot{\eta}_{kk}^{(\alpha)}\delta_{ij} + 2\mu_\alpha[\dot{\eta}_{ij}^{(\alpha)} - \dot{L}_{ij(0)}^{(\alpha)}] \tag{35}$$

$$\dot{\tau}^{(\alpha)}_{ij(1)} = \lambda_\alpha \dot{\xi}^{(\alpha)}_{kk}\delta_{ij} + 2\mu_\alpha[\dot{\xi}^{(\alpha)}_{ij} - \dot{L}^{(\alpha)}_{ij(1)}] \tag{36}$$

$$\dot{\tau}^{(\alpha)}_{ij(n)} = -2\mu_\alpha \dot{L}^{(\alpha)}_{ij(n)}, \qquad n = 2, \ldots, N \tag{37}$$

where $\lambda_\alpha = 2\nu_\alpha\mu_\alpha/[1-2\nu_\alpha]$.

Equations (35)–(37) form the stress–strain relations for each component $n$ in the series expansion of the stresses (24).

From (24) we obtain using (19) the following expression for the composite stresses utilizing the orthogonality properties of the Legendre polynomials.

$$(d_1 + d_2)\bar{\sigma}_{pj} = d_1\tau^{(1)}_{pj(0)} + d_2\tau^{(2)}_{pj(0)} \tag{38}$$

where $p = 1, 3$ and $j = 1, 2, 3$.

Similarly we obtain from (22) for the double stresses the expressions

$$M^{(\alpha)}_{p2j} = 2c_\alpha\tau^{(\alpha)}_{pj(1)}, \tag{39}$$

and for the layer stresses in (23)

$$\bar{\sigma}^{(\alpha)}_{2j} = \tau^{(\alpha)}_{2j(0)}. \tag{40}$$

Since the interface stresses $\Sigma_{2j}$ can be eliminated from (17) to (18), these equations after substituting (38)–(40) yield the field equations of motion of the elastic-viscoplastic composite modeled by the effective stiffness theory. It should be noticed that only the stresses $\tau^{(\alpha)}_{ij(0)}$, $\tau^{(\alpha)}_{ij(1)}$ enter into the equations of motion since the additional stresses $\tau^{(\alpha)}_{ij(n)}(n = 2, \ldots, N)$ perform no work upon the strains $\eta^{(\alpha)}_{ij}$, $\xi^{(\alpha)}_{ij}$.

The equations of motion for $\bar{u}^{(\alpha)}_j$ and $\Psi^{(\alpha)}_j$ can be expressed, using (35)–(36) and (15)–(16), in terms of the dependent variables $\bar{u}^{(\alpha)}_j$, $\Psi^{(\alpha)}_j$, $L^{(\alpha)}_{ij(0)}$ and $L^{(\alpha)}_{ij(1)}$. The evolution of the plastic strains is determined from the flow eqns (34).

In the following we apply the derived effective stiffness theory for the problem of a laminated slab subjected to a dynamic loading.

## APPLICATION TO A LAMINATED SLAB

We apply the previous theory to obtain the dynamic response of a laminated slab occupying the region $0 \le x_1 \le h$, $-\infty < x_2$, $x_3 < \infty$ and subjected to extended uniform time-dependent normal tractions on one of its surfaces ($x_1 = 0$), while its other surface ($x_1 = h$) is held rigidly clamped, see Fig. 2(b). Under this specific loading, from symmetry considerations it can be seen that eqns (10) will reduce to

$$u_1{}^{(\alpha)} = \bar{u}_1{}^{(\alpha)}(x_1, t),\ u_2{}^{(\alpha)} = \bar{x}_2{}^{(\alpha)}\Psi_2{}^{(\alpha)}(x_1, t),\ u_3{}^{(\alpha)} = 0. \tag{41}$$

From (11) $\bar{u}_1{}^{(\alpha)}(x_1, t) = \bar{u}_1(x_1, t)$ being the gross displacement and (13) gives

$$0 = \theta\Psi_2{}^{(1)}(x_1, t) + (1-\theta)\Psi_2{}^{(2)}(x_1, t). \tag{42}$$

The non-zero strain components, obtained from (15)–(16), are

$$\eta^{(\alpha)}_{11} = \partial_1\bar{u}_1,\ \eta^{(\alpha)}_{22} = \Psi_2{}^{(\alpha)},\ \xi^{(\alpha)}_{12} = \xi^{(\alpha)}_{21} = c_\alpha\partial_1\Psi_2{}^{(\alpha)}. \tag{43}$$

In the present plane-strain situation and stress–strain relations (35)–(36) simplify to

$$\left.\begin{aligned} \tau^{(\alpha)}_{ij(0)} &= \lambda_\alpha(\eta^{(\alpha)}_{11} + \eta^{(\alpha)}_{22})\delta_{ij} + 2\mu_\alpha(\eta^{(\alpha)}_{ij} - L^{(\alpha)}_{ij(0)}) \\ \tau^{(\alpha)}_{ij(1)} &= 2\mu_\alpha(\xi^{(\alpha)}_{ij} - L^{(\alpha)}_{ij(1)}) \end{aligned}\right\}\ i, j = 1, 2 \tag{44}$$

and

$$\tau^{(\alpha)}_{33(n)} = \nu_\alpha[\tau^{(\alpha)}_{11(n)} + \tau^{(\alpha)}_{22(n)}] + E_\alpha[L^{(\alpha)}_{11(n)} + L^{(\alpha)}_{22(n)}] \tag{45}$$

for $n = 0, 1, \ldots, N$.

The dynamic equations of motion (17) give now

$$\partial_1\bar{\sigma}_{11} = \rho\ddot{\bar{u}}_1 \tag{46}$$

whereas (18) give

$$\partial_1 M^{(\alpha)}_{122} + \Sigma_{22} - \bar{\sigma}^{(\alpha)}_{22} = I_\alpha \ddot{\Psi}^{(\alpha)}_2. \tag{47}$$

Equation (47) with $\alpha = 1, 2$ constitutes of two equations from which $\Sigma_{22}$ can be eliminated. This elimination gives using (42)

$$\partial_1[M^{(1)}_{122} - M^{(2)}_{122}] - [\bar{\sigma}^{(1)}_{22} - \bar{\sigma}^{(2)}_{22}] = c_3\ddot{\Psi}_2^{(1)} \tag{48}$$

where $c_3 = I_1 - I_2\theta/(\theta - 1)$.

From (38) and (43) we obtain the following expression for the composite stress

$$\bar{\sigma}_{11}(x_1, t) = c_4\partial_1\bar{u}_1 + c_5\Psi_2^{(1)} + c_6 L^{(1)}_{11(0)} + c_7 L^{(2)}_{11(0)} \tag{49}$$

where

$$c_4 = \theta(\lambda_1 + 2\mu_1) + (1 - \theta)(\lambda_2 + 2\mu_2)$$

$$c_5 = (\lambda_1 - \lambda_2)\theta, \quad c_6 = -2\theta\mu_1, \quad c_7 = 2(\theta - 1)\mu_2.$$

Consequently, (46) gives

$$c_4\partial_{11}\bar{u}_1 + c_5\partial_1\Psi_2^{(1)} + c_6\partial_1 L^{(1)}_{11(0)} + c_7\partial_1 L^{(2)}_{11(0)} = \rho\ddot{\bar{u}}_1 \tag{50}$$

which is the first equation of motion.

From (39) to (40) we have

$$M^{(\alpha)}_{122} = (d_\alpha^2/12)\partial_1\Psi_2^{(\alpha)} - 4c_\alpha\mu_\alpha L^{(\alpha)}_{12(1)} \tag{51}$$

$$\bar{\sigma}^{(\alpha)}_{22} = \lambda_\alpha\partial_1\bar{u}_1 + (\lambda_\alpha + 2\mu_\alpha)\Psi_2^{(\alpha)} - 2\mu_\alpha L^{(\alpha)}_{22(0)}. \tag{52}$$

Accordingly (47) yields the following second equation of motion

$$\begin{aligned} c_8\partial_{11}\Psi_2^{(1)} + (\lambda_2 - \lambda_1)\partial_1\bar{u}_1 - c_9\Psi_2^{(1)} \\ - 4[c_1\mu_1\partial_1 L^{(1)}_{12(1)} - c_2\mu_2\partial_1 L^{(2)}_{12(1)}] \\ + 2[\mu_1 L^{(1)}_{22(0)} - \mu_2 L^{(2)}_{22(0)}] = c_3\ddot{\Psi}_2^{(1)} \end{aligned} \tag{53}$$

where

$$c_8 = [d_1^2\mu_1 - d_2^2\mu_2\theta/(\theta - 1)]/12, \quad c_9 = \lambda_1 + 2\mu_1 - (\lambda_2 + 2\mu_2)\theta/(\theta - 1).$$

Equations (50) and (53) are two coupled non-linear differential equations in $\bar{u}_1$, $\Psi_2^{(1)}$ and the plastic strains $L^{(\alpha)}_{ijk(n)}$. The latter are governed by the flow rule (34).

In the special case of perfectly elastic constituents, $L^{(\alpha)}_{ijk(n)} \equiv 0$, and the equations become linear and can be easily shown that they reduce to those given, for example, by Thomas[11].

In Ref.[6] boundary conditions relating the external surface tractions and moments to the composite and double stresses are given. In the present case of normal traction on the boundary these conditions reduce to

$$\left.\begin{aligned} \sigma_{(1)j} &= \bar{\sigma}_{1j} \\ M^{(\alpha)}_{(1)2j} &= M_{12j} \end{aligned}\right\} \quad j = 1, 2, 3 \tag{54}$$

where, in terms of the actual tractions $\sigma_j^*$ on the boundary[12]

$$\sigma_{(1)j} = (\theta/d_1)\int_{-d_1/2}^{d_1/2} \sigma_j^*(x_2 + \bar{x}_2^{(1)})\,d\bar{x}_2^{(1)} + [(1-\theta)/d_2]\int_{-d_2/2}^{d_2/2} \sigma_j^*(x_2 + \bar{x}_2^{(2)})\,d\bar{x}_2^{(2)} \tag{55}$$

$$M^{(\alpha)}_{(1)2j} = (1/d_\alpha)\int_{-d_\alpha/2}^{d_\alpha/2} \sigma_j^*(x_2 + \bar{x}_2^{(\alpha)})\bar{x}_2^{(\alpha)}\,d\bar{x}_2^{(\alpha)}. \tag{56}$$

Under extended uniform time-dependent traction

$$\sigma_1^* = f(t), \quad \sigma_2^* = \sigma_3^* = 0. \tag{57}$$

Accordingly

$$\sigma_{(1)1} = f(t), \quad \sigma_{(1)2} = 0, \quad \sigma_{(1)3} = 0, \quad M^{(\alpha)}_{(1)2j} = 0. \tag{58}$$

Equations (58) together with the expressions for $M^{(\alpha)}_{12j}$ give on the loaded surface of the slab

$$\bar{\sigma}_{11}(x_1 = 0, t) = f(t) \tag{59}$$

$$\tau^{(\alpha)}_{1j(1)} = 0, \quad x_1 = 0. \tag{60}$$

From (60) we obtain for $j = 1$ and 2 that

$$L^{(\alpha)}_{11(1)} = 0, \quad \partial_1\Psi_2^{(\alpha)} = L^{(\alpha)}_{12(1)}, \quad x_1 = 0 \tag{61}$$

whereas for $j = 3$ the condition is satisfied identically. It can be easily verified that $L^{(\alpha)}_{12(n)}(x_1 = 0, t) = 0$ for all $n$ so that the second boundary condition in (61) reduces to $\partial_1\Psi_2^{(1)} = 0$ at $x_1 = 0$. From (25) we obtain also that $L^{(\alpha)}_{11(p)}(x_1 = 0, t) = 0$; $p = 1, \ldots, N$.

On the clamped surface we have

$$\bar{u}_1(x_1 = h, t) = 0, \quad \Psi_2^{(1)}(x_1 = h, t) = 0. \tag{62}$$

Finally, for a slab at rest prior to the application of the load at time $t = 0$, all the field variables are zero for $t < 0$.

## NUMERICAL SOLUTION

The numerical treatment of the field eqns (50), (53), (34) together with (59), (61) and (62) which govern the dynamic response of the laminated slab is based on a finite difference procedure. This is performed by dividing the interval $0 \le x_1 \le h$ into equal subintervals of size $\Delta x_1$ and introducing a time increment $\Delta t$. The procedure is essentially similar to that described in[13] for the solution of a two-dimensional dynamic problem of a homogeneous elastic-viscoplastic medium, and can be divided into three parts.

(1) In the first part the field variables $\bar{u}_1(x_1, t)$, $\psi_2^{(1)}(x_1, t)$ are computed at the internal points of the slab using a finite difference approximation to the equation of motion (50) and (53). This is obtained by approximating all the derivatives in those equations by their corresponding

central difference expressions giving an explicit three-level system of difference equations. Thus it is possible to compute the above field variables at time $t+\Delta t$ whenever their values at the time steps $t$-$\Delta t$ and $t$, and the values of the plastic strains $L^{(\alpha)}_{ij(n)} (n = 0, 1)$ at time $t$ are known throughout the region.

(2) In the second part the plastic strains $L^{(\alpha)}_{ij(n)}(x_1, t)$ are computed using the flow rule (34). The rate of the plastic work $\dot{w}_p{}^{(\alpha)}(x_1, t)$ is computed as in (26) but with the total strain rates $\dot{\epsilon}^{(\alpha)}_{ij}$ replaced by the plastic strain rates $\dot{\epsilon}^{(p\alpha)}_{ij}$ which are given in (3). Accordingly

$$\dot{w}_p{}^{(\alpha)} = t^{(\alpha)}_{ij(m)} t^{(\alpha)}_{ij(n)} F^{(\alpha)}_{mn} \tag{63}$$

using the definitions (29)–(30).

Equations (34) and (63) form the evolutionary equations used for the computation of $L^{(\alpha)}_{ij(n)}$ and $w_p{}^{(\alpha)}$ at time $t+\Delta t$ employing (29)–(30). The quantities $\Lambda_\alpha(x_1, t)$ in (30) are given by (4)–(7), and the integration in (30) over $-1 \le z^{(\alpha)} \le 1$ is approximated by using the trapezoidal method. Equations (34) and (63) are integrated by employing the improved Euler–Cauchy method according to which the field variables are first predicted tentatively at time $t+\Delta t$ and then corrected using the predicted values.

(3) This part consists of the computation of $\bar{u}_1(x_1, t)$, $\psi_2{}^{(1)}(x_1, t)$ at the slab surfaces $x_1 = 0, h$ using the boundary conditions (59) and (61) to (62).

## RESULTS

We choose to illustrate the preceding theory by presenting the dynamic response of a laminated slab made of titanium ($\alpha = 1$) and copper ($\alpha = 2$). The appropriate parameters of these materials are given by [10, 14]:

$$\nu_1 = 0.34, \quad \mu_1 = 0.44 \times 10^{11}\ \text{N/m}^2, \quad D_0{}^{(1)} = 10^4\ \text{sec}^{-1}, \quad n_1 = 1,$$
$$Z_0{}^{(1)} = 1.15 \times 10^9\ \text{N/m}^2, \quad Z_1{}^{(1)} = 1.4 \times 10^9\ \text{N/m}^2, \quad m_1 = 100,$$
$$\rho_1 = 4.87 \times 10^3\ \text{Kg/m}^3.$$
$$\nu_2 = 0.33, \quad \mu_2 = 0.45 \times 10^{11}\ \text{N/m}^2, \quad D_0{}^{(2)} = 10^4\ \text{sec}^{-1}, \quad n_2 = 7.5,$$
$$Z_0{}^{(2)} = 0.63 \times 10^8\ \text{N/m}^2, \quad Z_1{}^{(2)} = 2.5 \times 10^8\ \text{N/m}^2, \quad m_2 = 8.19,$$
$$\rho_2 = 8.96 \times 10^3\ \text{Kg/m}^3.$$

It should be noticed that the elastic parameters of the materials are almost identical, but there is a considerable contrast between the constants which describe their plastic behavior.

As a reference length we use $d = d_1 + d_2$ which is chosen such that a compressional elastic wave whose speed in the titanium is $c_0 = [(\lambda_1 + 2\mu_1)/\rho_1]^{1/2}$ will propagate a distance $d$ during the time interval $1/D_0{}^{(1)}$. For the reinforcement ratio we choose $\theta = 1/2$, and the width of the slab $h/d = 0.2$. The spatial increment is taken as $\Delta x_1/d = 0.02$ and the temporal increment $c_0\Delta t/d = 0.005$ which is found to give satisfactory accuracy. The range of integration $-1 \le z^{(\alpha)} \le 1$ in (30) was divided into 20 intervals which also yield satisfactory accurate results.

The applied input on the surface $x_1 = 0$ is described by $f(t)$ in (59). This is taken to have either the form

$$f(t) = \begin{cases} f_0 & \sin(\pi t/2t_m) \quad t \le t_m \\ f_0 & \quad t > t_m \end{cases} \tag{64}$$

or

$$f(t) = \begin{cases} f_0 & \sin(2\pi t/t_m) \quad t \le t_m \\ 0 & \quad t > t_m \end{cases} \tag{65}$$

with $f_0$, $t_m$ being appropriate constants. The temporal behavior (64) describes a loading input, whereas (65) corresponds to a loading–unloading applied load on $x_1 = 0$.

(1) *Loading input*

In this case $f(t)$ is given by (64) with $f_0/(\lambda_1+2\mu_1)=0.02$ and $c_0t_m/d=0.1$. In Fig. 2 the composite stress $\bar{\sigma}_{11}(x_1, t)$ (given by (49)) is shown at the midpoint $x_1=h/2$ versus the time $c_0t/d$ for $N=5$ terms. It turns out that with $N=4$ terms some inaccuracies appear in the plot which are eliminated by taking $N=5$ terms and additional terms show no further variations. In the same figure we also present for comparison the resulting composite stress in a laminated slab made of titanium and copper assumed to be perfectly elastic. The basic effect of the viscoplastic mechanism in the spreading and attenuation of the propagating pulse is well exhibited in this figure. It should be noticed that the dynamic response exhibited in the figure contains both the direct and reflected waves.

(2) *Loading–unloading input*

For a loading–unloading input, $f(t)$ is given by (65) with $f_0/(\lambda_1+2\mu_1)=0.02$ and $c_0t_m/d=0.4$. The resulting composite stress $\bar{\sigma}_{11}$ is shown in Fig. 3 at the observation point $x_1=h/2$ versus the time (with $N=5$ terms). In this case there is an unloading phenomenon caused directly by the applied input. The comparison with the composite stress excited in the corresponding perfectly elastic laminated slab is also shown in the Figure exhibiting the effects of viscoplasticity in the present situation.

We conclude this section with the following remark. The unified theory of viscoplasticity given by (3)–(8) is applied here in impact loading conditions. It should be mentioned that the validity of this unified theory was examined in dynamic situations in [15] where the response of elastic–viscoplastic beams subjected to dynamic loads was computed. Comparsons between theoretical and experimental dynamic deflections show good agreement for relatively short response times.

## CONCLUSIONS

A first order effective stiffness theory is presented for modeling the dynamic behavior of a laminated composite made of periodic elastic–viscoplastic constituents. Each material is represented by a unified theory of viscoplasticity in which no yield condition or loading or unloading are required. The theory is applied for a laminated slab subjected to time-dependent loadings.

A generalization to a second order theory in which the displacements are represented by a quadratic distributions [5] rather than linear (10) is possible. Extension of the theory to anisotropic layers can be also made.

The developed effective stiffness theory can be essentially applied to other types of inelastic materials as long as their constitutive relations can be represented in a sutiable unified form.

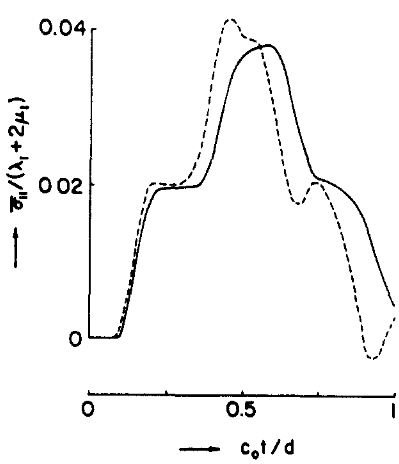


Fig. 2.

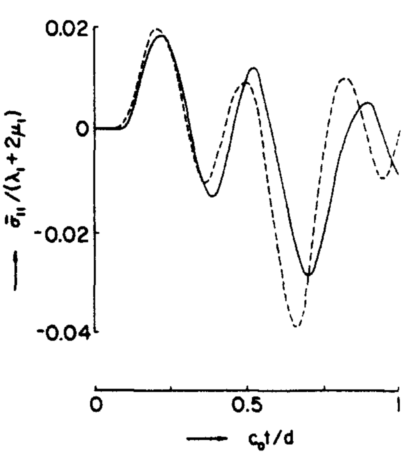


Fig. 3.

Fig. 2. The composite stress $\bar{\sigma}_{11}$ versus time at the mid-point $x_1=h/2$ of a laminated elastic–viscoplastic slab (solid line) and a laminated perfectly elastic slab (dashed line). The input is given by (64).

Fig. 3. Same as Fig. 2 but for the input (65).

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